On the average indices of closed geodesics on positively curved Finsler spheres

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Abstract

In this paper, we prove that on every Finsler n-sphere (S^n, F) for $n \geq 6$ with reversibility λ and flag curvature K satisfying $\left(\frac{\lambda}{\lambda+1}\right)^2 < K \leq 1$, either there exist infinitely many prime closed geodesics or there exist $[\frac{n}{2}] - 2$ closed geodesics possessing irrational average indices. If in addition the metric is bumpy, then there exist $n - 3$ closed geodesics possessing irrational average indices provided the number of closed geodesics is finite.

Key words: Finsler spheres, closed geodesics, index iteration, average index.

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Running head: Closed geodesics on Finsler spheres

1 Introduction and main results

This paper is devoted to a study on closed geodesics on Finsler n-spheres. Let us recall firstly the definition of the Finsler metrics.

Definition 1.1. (cf. [She1]) *Let M be a finite dimensional manifold. A function $F : TM \to [0, +\infty)$ is a* Finsler metric *if it satisfies*

(F1) F is C^∞ on $TM \setminus \{0\}$,

(F2) $F(x, \lambda y) = \lambda F(x, y)$ for all $y \in T_x M$, $x \in M$, and $\lambda > 0$,

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(F3) For every $y \in T_xM \setminus \{0\}$, the quadratic form

$$g_{x,y}(u,v) \equiv \frac{1}{2}\frac{\partial^2}{\partial s \partial t}F^2(x, y + su + tv)|_{t=s=0}, \qquad \forall u, v \in T_xM,$$

is positive definite.

In this case, (M, F) is called a Finsler manifold. F is reversible if $F(x, -y) = F(x, y)$ holds for all $y \in T_xM$ and $x \in M$. F is Riemannian if $F(x, y)^2 = \frac{1}{2}G(x)y \cdot y$ for some symmetric positive definite matrix function $G(x) \in GL(T_xM)$ depending on $x \in M$ smoothly.

A closed curve in a Finsler manifold is a closed geodesic if it is locally the shortest path connecting any two nearby points on this curve (cf. [She1]). As usual, on any Finsler n-sphere $S^n = (S^n, F)$, a closed geodesic $c : S^1 = \mathbf{R}/\mathbf{Z} \to S^n$ is *prime* if it is not a multiple covering (i.e., iteration) of any other closed geodesics. Here the m-th iteration c^m of c is defined by $c^m(t) = c(mt)$. The inverse curve c^{-1} of c is defined by $c^{-1}(t) = c(1 - t)$ for $t \in \mathbf{R}$. We call two prime closed geodesics c and d *distinct* if there is no $\theta \in (0, 1)$ such that $c(t) = d(t + \theta)$ for all $t \in \mathbf{R}$. We shall omit the word *distinct* when we talk about more than one prime closed geodesic. On a symmetric Finsler (or Riemannian) n-sphere, two closed geodesics c and d are called *geometrically distinct* if $c(S^1) \neq d(S^1)$, i.e., their image sets in S^n are distinct.

For a closed geodesic c on (S^n, F), denote by P_c the linearized Poincaré map of c (cf. p.143 of [Zil1]). Then $P_c \in \mathrm{Sp}(2n - 2)$ is a symplectic matrix. For any $M \in \mathrm{Sp}(2k)$, we define the *elliptic height* $e(M)$ of M to be the total algebraic multiplicity of all eigenvalues of M on the unit circle $\mathbf{U} = \{z \in \mathbf{C}| \ |z| = 1\}$ in the complex plane $\mathbf{C}$. Since M is symplectic, $e(M)$ is even and $0 \leq e(M) \leq 2k$. Then c is called *hyperbolic* if all the eigenvalues of P_c avoid the unit circle in $\mathbf{C}$, i.e., $e(P_c) = 0$; *elliptic* if all the eigenvalues of P_c are on the unit circle, i.e., $e(P_c) = 2(n - 1)$. Recall that a Finsler metric F is *bumpy* if all the closed geodesics on (S^n, F) are non-degenerate, i.e., $1 \notin \sigma(P_c)$ for any closed geodesic c.

Following H-B. Rademacher in [Rad4], the reversibility $\lambda = \lambda(M, F)$ of a compact Finsler manifold (M, F) is defied to be

$$\lambda := \max\{F(-X) \,|\, X \in TM, F(X) = 1\} \geq 1.$$

We are aware of a number of results concerning closed geodesics on spheres. In [Fet1] of 1965, A. Fet proved that every bumpy Riemannian metric on a simply connected compact manifold carries at least two geometrically distinct closed geodesics. Motivated by the work [Kli1] of W. Klingenberg in 1969, W. Ballmann, G. Thorbergsson and W. Ziller studied in [BTZ1] and [BTZ2] of 1982-83 the existence and stability of closed geodesics on positively curved compact rank one symmetric spaces

under pinching conditions. In [Hin1] of 1984, N. Hingston proved that a Riemannian metric on a sphere all of whose closed geodesics are hyperbolic carries infinitely many geometrically distinct closed geodesics. By the results of J. Franks in [Fra1] of 1992 and V. Bangert in [Ban1] of 1993, there are infinitely many geometrically distinct closed geodesics for any Riemannian metric on S^2.

It was quite surprising when A. Katok [Kat1] in 1973 found some non-symmetric Finsler metrics on S^n with only finitely many prime closed geodesics and all closed geodesics are non-degenerate and elliptic. In Katok's examples the spheres S^{2n} and S^{2n-1} have precisely $2n$ closed geodesics (cf. also [Zil1]). In [Rad5], H.-B. Rademacher studied the existence and stability of closed geodesics on positively curved Finsler manifolds. In a recent paper of V. Bangert and Y. Long [BaL1], they proved that on any Finsler 2-sphere (S^2, F), there exist at least two prime closed geodesics.

The following are the main results in this paper:

Theorem 1.2. *On every Finsler n-sphere (S^n, F) for $n \geq 6$ with reversibility λ and flag curvature K satisfying $\left(\frac{\lambda}{\lambda+1}\right)^2 < K \leq 1$, either there exist infinitely many prime closed geodesics or there exist $[\frac{n}{2}] - 2$ closed geodesics possessing irrational average indices.*

Theorem 1.3. *On every bumpy Finsler n-sphere (S^n, F) for $n \geq 4$ with reversibility λ and flag curvature K satisfying $\left(\frac{\lambda}{\lambda+1}\right)^2 < K \leq 1$, there exist $n - 3$ closed geodesics possessing irrational average indices provided the number of closed geodesics is finite.*

Remark 1.4. Note that on the standard Riemannian n-sphere of constant curvature 1, all geodesics are closed and their average indices are integers. Thus one can not hope that Theorems 1.2 and 1.3 hold for all Finsler n-spheres. Note also that in [LoW1] od Y. Long and the author, they proved the existence of at least two prime closed geodesics possessing irrational average indices on every Finsler 2-sphere (S^2, F) provided the number of prime closed geodesics is finite by a completely different method.

The proof of these theorems is motivated by Theorem 1.3 in [LoZ1]. In this paper, we use the Fadell-Rabinowitz index theory in a relative version to obtain the desired critical values of the energy functional E on the space pair (Λ, Λ^0), where Λ is the free loop space of S^n and Λ^0 is its subspace consisting of constant point curves. Then we use the method of index iteration theory of Symplectic paths developed by Y. Long and his coworkers, especially the common index jump theorem to obtain the desired results.

In this paper, let $\mathbf{N}$, $\mathbf{N}_0$, $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{C}$ denote the sets of natural integers, non-negative integers, integers, rational numbers, real numbers, and complex numbers respectively. We use only singular homology modules with $\mathbf{Q}$-coefficients. For an S^1-space X, we denote by $\overline{X}$ the quotient space X/S^1. We denote by $[a] = \max\{k \in \mathbf{Z} \,|\, k \leq a\}$ for any $a \in \mathbf{R}$.

2   Critical point theory for closed geodesics

In this section, we will study critical point theory for closed geodesics.

On a compact Finsler manifold (M, F), we choose an auxiliary Riemannian metric. This endows the space $\Lambda = \Lambda M$ of H^1-maps $\gamma : S^1 \to M$ with a natural Riemannian Hilbert manifold structure on which the group $S^1 = \mathbf{R}/\mathbf{Z}$ acts continuously by isometries, cf. [Kli2], Chapters 1 and 2. This action is defined by translating the parameter, i.e.,

$$(s \cdot \gamma)(t) = \gamma(t + s)$$

for all $\gamma \in \Lambda$ and $s, t \in S^1$. The Finsler metric F defines an energy functional E and a length functional L on Λ by

$$E(\gamma) = \frac{1}{2} \int_{S^1} F(\dot{\gamma}(t))^2 dt, \quad L(\gamma) = \int_{S^1} F(\dot{\gamma}(t)) dt. \tag{2.1}$$

Both functionals are invariant under the S^1-action. By [Mer1], the functional E is $C^{1,1}$ on Λ and satisfies the Palais-Smale condition. Thus we can apply the deformation theorems in [Cha1] and [MaW1]. The critical points of E of positive energies are precisely the closed geodesics $c : S^1 \to M$ of the Finsler structure. If $c \in \Lambda$ is a closed geodesic then c is a regular curve, i.e. $\dot{c}(t) \neq 0$ for all $t \in S^1$, and this implies that the second differential $E''(c)$ of E at c exists. As usual we define the index $i(c)$ of c as the maximal dimension of subspaces of $T_c \Lambda$ on which $E''(c)$ is negative definite, and the nullity $\nu(c)$ of c so that $\nu(c) + 1$ is the dimension of the null space of $E''(c)$.

For $m \in \mathbf{N}$ we denote the m-fold iteration map $\phi^m : \Lambda \to \Lambda$ by

$$\phi^m(\gamma)(t) = \gamma(mt) \qquad \forall \gamma \in \Lambda, t \in S^1. \tag{2.2}$$

We also use the notation $\phi^m(\gamma) = \gamma^m$. For a closed geodesic c, the average index is defined by

$$\hat{i}(c) = \lim_{m \to \infty} \frac{i(c^m)}{m}. \tag{2.3}$$

If $\gamma \in \Lambda$ is not constant then the multiplicity $m(\gamma)$ of γ is the order of the isotropy group $\{s \in S^1 \mid s \cdot \gamma = \gamma\}$. If $m(\gamma) = 1$ then γ is called *prime*. Hence $m(\gamma) = m$ if and only if there exists a prime curve $\tilde{\gamma} \in \Lambda$ such that $\gamma = \tilde{\gamma}^m$.

In this paper for $\kappa \in \mathbf{R}$ we denote by

$$\Lambda^\kappa = \{d \in \Lambda \mid E(d) \leq \kappa\}, \tag{2.4}$$

For a closed geodesic c we set

$$\Lambda(c) = \{\gamma \in \Lambda \mid E(\gamma) < E(c)\}.$$

If $A \subseteq \Lambda$ is invariant under some subgroup Γ of S^1, we denote by A/Γ the quotient space of A with respect to the action of Γ. Using singular homology with rational coefficients we will consider the following critical $\mathbf{Q}$-module of a closed geodesic $c \in \Lambda$:

$$\overline{C}_*(E, c) = H_* \left((\Lambda(c) \cup S^1 \cdot c)/S^1, \Lambda(c)/S^1 \right). \tag{2.5}$$

Following [Rad2], Section 6.2, we can use finite-dimensional approximations to Λ to apply the results of D. Gromoll and W. Meyer [GrM1] to a given closed geodesic c which is isolated as a critical orbit. Then we have

Proposition 2.1. *Let $k_j(c) \equiv \dim \overline{C}_j(E, c)$. Then $k_j(c)$ equal to 0 when $j < i(c)$ or $j > i(c) + \nu(c)$ and can only take values 0 or 1 when $j = i(c)$ or $j = i(c) + \nu(c)$.*

Next we recall the Fadell-Rabinowitz index in a relative version due to [Rad3]. Let X be an S^1-space, $A \subset X$ a closed S^1-invariant subset. Note that the cup product defines a homomorphism

$$H_{S^1}^*(X) \otimes H_{S^1}^*(X,\ A) \rightarrow H_{S^1}^*(X,\ A) : \quad (\zeta,\ z) \rightarrow \zeta \cup z, \tag{2.6}$$

where $H_{S^1}^*$ is the S^1-equivariant cohomology with rational coefficients in the sense of A. Borel (cf. Chapter IV of [Bor1]). We fix a characteristic class $\eta \in H^2(CP^\infty)$. Let $f^* : H^*(CP^\infty) \rightarrow H_{S^1}^*(X)$ be the homomorphism induced by a classifying map $f : X_{S^1} \rightarrow CP^\infty$. Now for $\gamma \in H^*(CP^\infty)$ and $z \in H_{S^1}^*(X,\ A)$, let $\gamma \cdot z = f^*(\gamma) \cup z$. Then the order $ord_\eta(z)$ with respect to η is defined by

$$ord_\eta(z) = \inf\{k \in \mathbf{N} \cup \{\infty\} \mid \eta^k \cdot z = 0\}. \tag{2.7}$$

By Proposition 3.1 of [Rad3], there is an element $z \in H_{S^1}^{n+1}(\Lambda,\ \Lambda^0)$ of infinite order, i.e., $ord_\eta(z) = \infty$. For $\kappa \geq 0$, we denote by $j_\kappa : (\Lambda^\kappa,\ \Lambda^0) \rightarrow (\Lambda\ \Lambda^0)$ the natural inclusion and define the function $d_z : \mathbf{R}^{\geq 0} \rightarrow \mathbf{N} \cup \{\infty\}$:

$$d_z(\kappa) = ord_\eta(j_\kappa^*(z)). \tag{2.8}$$

Denote by $d_z(\kappa-) = \lim_{\epsilon \searrow 0} d_z(\kappa - \epsilon)$, where $t \searrow a$ means $t > a$ and $t \rightarrow a..$

Then we have the following property due to Section 5 of [Rad3]

Lemma 2.2. (H.-B. Rademacher) *The function d_z is non-decreasing and $\lim_{\lambda \searrow \kappa} d_z(\lambda) = d_z(\kappa)$. Each discontinuous point of d_z is a critical value of the energy functional E. In particular, if $d_z(\kappa) - d_z(\kappa-) \geq 2$, then there are infinitely many prime closed geodesics c with energy κ.* ∎

For each $i \geq 1$, we define

$$\kappa_i = \inf\{\delta \in \mathbf{R} \mid d_z(\delta) \geq i\}. \tag{2.9}$$

Then we have the following.

Lemma 2.3. *Suppose there are only finitely many prime closed geodesics on (S^n, F). Then each κ_i is a critical value of E. If $\kappa_i = \kappa_j$ for some $i < j$, then there are infinitely many prime closed geodesics on (S^n, F).*

Proof. It follows from the S^1-equivariant deformation theorem (cf. Theorem 1.7.2 of [Cha1]) that each κ_i is a critical value of E. Now suppose $\kappa_i = \kappa_j$ for some $i < j$. Then by (2.9), we have $d_z(\kappa_i-) < i$ and $d_z(\kappa_i) = d_z(\kappa_j) \geq j \geq d_z(\kappa_i-) + 2$. Hence we have $d_z(\kappa_i) - d_z(\kappa_i-) \geq 2$. Thus Lemma 2.2 implies there are infinitely many prime closed geodesics c with energy κ_i. This proves the lemma. ∎

Lemma 2.4. *Suppose there are only finitely many prime closed geodesics on (S^n, F). Then for every $i \in \mathbf{N}$, there exists a closed geodesic c on (S^n, F) such that*

$$E(c) = \kappa_i, \quad \overline{C}_{2i+\dim(z)-2}(E, c) \neq 0. \tag{2.10}$$

Proof. By (2.8), we have $d_z(\epsilon) = 0$ for any $\epsilon > 0$ sufficiently small. This holds since Λ^0 is a strong deformation retract of Λ^ϵ for $\epsilon > 0$ sufficiently small (cf. Theorem 1.4.15 of [Kli2]), and then $j_\epsilon^*(z) = 0$. Thus it follows from Lemma 2.3 that $d_z(\kappa_i) = i$. Hence it follows from Lemma 5.8 of [Rad3] that

$$H_{S^1}^{2i+\dim(z)-2}(\Lambda^{\kappa_i+\epsilon}, \Lambda^{\kappa_i-\epsilon}) \neq 0, \tag{2.11}$$

for any $\epsilon > 0$ sufficiently small.

Since any $\gamma \in \Lambda^{\kappa_i+\epsilon} \setminus \Lambda^{\kappa_i-\epsilon}$ is not a fixed point of the S^1-action, its isotropy group is finite. Hence we can use Lemma 6.11 of [FaR1] to obtain

$$H_{S^1}^*(\Lambda^{\kappa_i+\epsilon}, \Lambda^{\kappa_i-\epsilon}) \cong H^*(\Lambda^{\kappa_i+\epsilon}/S^1, \Lambda^{\kappa_i-\epsilon}/S^1). \tag{2.12}$$

By the finiteness assumption of the number of prime closed geodesics, a small perturbation on the energy functional can be applied to reduce each critical orbit to nearby non-degenerate ones. Thus similar to the proofs of Lemma 2 of [GrM1] and Lemma 4 of [GrM2], all the homological $\mathbf{Q}$-modules of $(\Lambda^{\kappa_i+\epsilon}, \Lambda^{\kappa_i-\epsilon})$ is finitely generated. Therefore we can apply Theorem 5.5.3 and Corollary 5.5.4 on pages 243-244 of [Spa1] to obtain

$$H_*(\Lambda^{\kappa_i+\epsilon}/S^1, \Lambda^{\kappa_i-\epsilon}/S^1) \cong H^*(\Lambda^{\kappa_i+\epsilon}/S^1, \Lambda^{\kappa_i-\epsilon}/S^1). \tag{2.13}$$

By Theorem 1.4.2 of [Cha1], we have

$$H_*(\Lambda^{\kappa_i+\epsilon}/S^1, \Lambda^{\kappa_i-\epsilon}/S^1) = \bigoplus_{E(c)=\kappa_i} \overline{C}_*(E, c). \tag{2.14}$$

Now our lemma follows from (2.11)-(2.14). ∎

Definition 2.5. *A prime closed geodesic c is (m,i)-* **variationally visible:** *if there exist some $m, i \in \mathbf{N}$ such that (2.10) holds for c^m and κ_i. We call c* **infinitely variationally visible:** *if there exist infinitely many $m, i \in \mathbf{N}$ such that c is (m,i)-variationally visible. We denote by $\mathcal{V}_\infty(S^n, F)$ the set of infinitely variationally visible closed geodesics.*

Theorem 2.6. *Suppose there are only finitely many prime closed geodesics on (S^n, F). Then for any $c \in \mathcal{V}_\infty(S^n, F)$, we have*

$$\frac{\hat{i}(c)}{L(c)} = 2\sigma. \tag{2.15}$$

where $\sigma = \liminf_{i \to \infty} i/\sqrt{2\kappa_i} = \limsup_{i \to \infty} i/\sqrt{2\kappa_i}$.

Proof. Note that we have $\hat{i}(c^m) = m\hat{i}(c)$ by (2.3) and $L(c^m) = mL(c)$. Thus $\frac{\hat{i}(c^m)}{L(c^m)} = \frac{\hat{i}(c)}{L(c)}$ for any $m \in \mathbf{N}$. Now the lemma follows from Lemmas 5.12, 6.1 and Corollary 6.3 of [Rad3]. ∎

3 Index iteration theory for closed geodesics

Let c be a closed geodesic on a Finsler n-sphere $S^n = (S^n, F)$. Denote the linearized Poincaré map of c by $P_c \in \mathrm{Sp}(2n - 2)$. Then P_c is a symplectic matrix. Note that the index iteration formulae in [Lon3] of 2000 (cf. Chap. 8 of [Lon4]) work for Morse indices of iterated closed geodesics (cf. [LLo1], Chap. 12 of [Lon4]). Since every closed geodesic on a sphere must be orientable. Then by Theorem 1.1 of [Liu1] of C. Liu (cf. also [Wil1]), the initial Morse index of a closed geodesic c on a n-dimensional Finsler sphere coincides with the index of a corresponding symplectic path introduced by C. Conley, E. Zehnder, and Y. Long in 1984-1990 (cf. [Lon4]).

Note that the precise index iteration formulae of Y. Long (cf. Theorem 8.3.1 of [Lon4]) is established upon the decomposition of the end matrix $\gamma(\tau)$ of the symplectic path $\gamma : [0, \tau] \to \mathrm{Sp}(2n)$ within $\Omega^0(\gamma(\tau))$ in Theorem 1.8.10 and the first part of Theorem 8.3.1 of [Lon4], which leads to the 2×2 or 4×4 basic normal form decomposition of $\gamma(\tau)$. Specially it is proved in Lemma 9.1.5 of [Lon4] that the splitting numbers of M are constants on $\Omega^0(M)$, where

$$\Omega(M) = \{N \in \mathrm{Sp}(2n) \quad | \quad \sigma(N) \cap \mathbf{U} = \sigma(M) \cap \mathbf{U},$$
$$\dim_{\mathbf{C}} \ker_{\mathbf{C}}(N - \lambda I) = \dim_{\mathbf{C}} \ker_{\mathbf{C}}(M - \lambda I), \ \forall \lambda \in \sigma(M) \cap \mathbf{U}\},$$

where $\mathbf{U} = \{z \in \mathbf{C} \,|\, |z| = 1\}$. $\Omega^0(M)$ is defined to be the path connected component of $\Omega(M)$ which contains M. The Bott iteration formulae in [Bot1] and [BTZ1] are based on decomposition of the end matrix $\gamma(\tau)$ of the symplectic path $\gamma : [0, \tau] \to \mathrm{Sp}(2n)$ within $[\gamma(\tau)]$, the conjugate set of $\gamma(\tau)$. Specially it is proved that the splitting numbers of M in [Bot1] and [BTZ1] are constants

on $[M] \equiv \{P^{-1}MP \,|\, P \in \mathrm{Sp}(2n)\}$. Note that $[M]$ is a proper subset of $\Omega^0(M)$ in general for $M \in \mathrm{Sp}(2n)$. Note also that there are only 11 basic normal forms (cf. [Lon4]), and they are only 2×2 or 4×4 matrices. Thus they are simpler than usual normal forms, and then it is possible to use different patterns of the iteration formula Theorem 8.3.1 of [Lon4] to classify symplectic paths as well as closed geodesics to carry out proofs. This is a major difference between formulae established in [Lon3] and Bott-type formulae established in [Bot1], [BTZ1] and in [Lon2]. Hence in this section, we recall briefly the index theory for symplectic paths. All the details can be found in [Lon4].

As usual, the symplectic group $\mathrm{Sp}(2n)$ is defined by

$$\mathrm{Sp}(2n) = \{M \in \mathrm{GL}(2n, \mathbf{R}) \,|\, M^T JM = J\},$$

whose topology is induced from that of $\mathbf{R}^{4n^2}$, where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ and I_n is the identity matrix in $\mathbf{R}^n$. For $\tau > 0$ we are interested in paths in $\mathrm{Sp}(2n)$:

$$\mathcal{P}_\tau(2n) = \{\gamma \in C([0, \tau], \mathrm{Sp}(2n)) \,|\, \gamma(0) = I_{2n}\},$$

which is equipped with the topology induced from that of $\mathrm{Sp}(2n)$. The following real function was introduced in [Lon2]:

$$D_\omega(M) = (-1)^{n-1}\overline{\omega}^n \det(M - \omega I_{2n}), \qquad \forall \omega \in \mathbf{U}, \; M \in \mathrm{Sp}(2n).$$

Thus for any $\omega \in \mathbf{U}$ the following codimension 1 hypersurface in $\mathrm{Sp}(2n)$ is defined in [Lon2]:

$$\mathrm{Sp}(2n)_\omega^0 = \{M \in \mathrm{Sp}(2n) \,|\, D_\omega(M) = 0\}.$$

For any $M \in \mathrm{Sp}(2n)_\omega^0$, we define a co-orientation of $\mathrm{Sp}(2n)_\omega^0$ at M by the positive direction $\frac{d}{dt} Me^{t\epsilon J}|_{t=0}$ of the path $Me^{t\epsilon J}$ with $0 \le t \le 1$ and $\epsilon > 0$ being sufficiently small. Let

$$
\begin{aligned}
\mathrm{Sp}(2n)_\omega^* &= \mathrm{Sp}(2n) \setminus \mathrm{Sp}(2n)_\omega^0, \\
\mathcal{P}_{\tau,\omega}^*(2n) &= \{\gamma \in \mathcal{P}_\tau(2n) \,|\, \gamma(\tau) \in \mathrm{Sp}(2n)_\omega^*\}, \\
\mathcal{P}_{\tau,\omega}^0(2n) &= \mathcal{P}_\tau(2n) \setminus \mathcal{P}_{\tau,\omega}^*(2n).
\end{aligned}
$$

For any two continuous arcs ξ and $\eta : [0, \tau] \to \mathrm{Sp}(2n)$ with $\xi(\tau) = \eta(0)$, it is defined as usual:

$$
\eta * \xi(t) = \begin{cases} \xi(2t), & \text{if } 0 \le t \le \tau/2, \\ \eta(2t - \tau), & \text{if } \tau/2 \le t \le \tau. \end{cases}
$$

Given any two $2m_k \times 2m_k$ matrices of square block form $M_k = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix}$ with $k = 1, 2$, as in [Lon4], the $\diamond$-product of M_1 and M_2 is defined by the following $2(m_1 + m_2) \times 2(m_1 + m_2)$ matrix $M_1 \diamond M_2$:

$$M_1 \diamond M_2 = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}.$$

Denote by $M^{\diamond k}$ the k-fold $\diamond$-product $M \diamond \cdots \diamond M$. Note that the $\diamond$-product of any two symplectic matrices is symplectic. For any two paths $\gamma_j \in \mathcal{P}_\tau(2n_j)$ with $j = 0$ and 1, let $\gamma_0 \diamond \gamma_1(t) = \gamma_0(t) \diamond \gamma_1(t)$ for all $t \in [0, \tau]$.

A special path $\xi_n \in \mathcal{P}_\tau(2n)$ is defined by

$$\xi_n(t) = \begin{pmatrix} 2 - \frac{t}{\tau} & 0 \\ 0 & (2 - \frac{t}{\tau})^{-1} \end{pmatrix}^{\diamond n} \qquad \text{for } 0 \leq t \leq \tau. \tag{3.1}$$

Definition 3.1. (cf. [Lon2], [Lon4]) *For any $\omega \in \mathbf{U}$ and $M \in \mathrm{Sp}(2n)$, define*

$$\nu_\omega(M) = \dim_{\mathbf{C}} \ker_{\mathbf{C}}(M - \omega I_{2n}). \tag{3.2}$$

For any $\tau > 0$ and $\gamma \in \mathcal{P}_\tau(2n)$, define

$$\nu_\omega(\gamma) = \nu_\omega(\gamma(\tau)). \tag{3.3}$$

If $\gamma \in \mathcal{P}_{\tau,\omega}^*(2n)$, define
$$i_\omega(\gamma) = [\mathrm{Sp}(2n)_\omega^0 : \gamma * \xi_n], \tag{3.4}$$

where the right hand side of (3.4) is the usual homotopy intersection number, and the orientation of $\gamma * \xi_n$ is its positive time direction under homotopy with fixed end points.

If $\gamma \in \mathcal{P}_{\tau,\omega}^0(2n)$, we let $\mathcal{F}(\gamma)$ be the set of all open neighborhoods of γ in $\mathcal{P}_\tau(2n)$, and define

$$i_\omega(\gamma) = \sup_{U \in \mathcal{F}(\gamma)} \inf\{i_\omega(\beta) \mid \beta \in U \cap \mathcal{P}_{\tau,\omega}^*(2n)\}. \tag{3.5}$$

Then

$$(i_\omega(\gamma), \nu_\omega(\gamma)) \in \mathbf{Z} \times \{0, 1, \ldots, 2n\},$$

is called the index function of γ at ω.

Note that when $\omega = 1$, this index theory was introduced by C. Conley-E. Zehnder in [CoZ1] for the non-degenerate case with $n \geq 2$, Y. Long-E. Zehnder in [LZe1] for the non-degenerate case with $n = 1$, and Y. Long in [Lon1] and C. Viterbo in [Vit1] independently for the degenerate case.

The case for general $\omega \in \mathbf{U}$ was defined by Y. Long in [Lon2] in order to study the index iteration theory (cf. [Lon4] for more details and references).

For any symplectic path $\gamma \in \mathcal{P}_\tau(2n)$ and $m \in \mathbf{N}$, we define its m-th iteration $\gamma^m : [0, m\tau] \to \mathrm{Sp}(2n)$ by

$$\gamma^m(t) = \gamma(t - j\tau)\gamma(\tau)^j, \qquad \text{for} \quad j\tau \le t \le (j+1)\tau, \ j = 0, 1, \ldots, m-1. \tag{3.6}$$

We still denote the extended path on $[0, +\infty)$ by γ.

Definition 3.2. (cf. [Lon2], [Lon4]) *For any $\gamma \in \mathcal{P}_\tau(2n)$, we define*

$$(i(\gamma, m), \nu(\gamma, m)) = (i_1(\gamma^m), \nu_1(\gamma^m)), \qquad \forall m \in \mathbf{N}. \tag{3.7}$$

The mean index $\hat{i}(\gamma, m)$ per $m\tau$ for $m \in \mathbf{N}$ is defined by

$$\hat{i}(\gamma, m) = \lim_{k \to +\infty} \frac{i(\gamma, mk)}{k}. \tag{3.8}$$

For any $M \in \mathrm{Sp}(2n)$ and $\omega \in \mathbf{U}$, the splitting numbers $S_M^\pm(\omega)$ of M at ω are defined by

$$S_M^\pm(\omega) = \lim_{\epsilon \to 0^+} i_{\omega \exp(\pm\sqrt{-1}\epsilon)}(\gamma) - i_\omega(\gamma), \tag{3.9}$$

for any path $\gamma \in \mathcal{P}_\tau(2n)$ satisfying $\gamma(\tau) = M$.

For a given path $\gamma \in \mathcal{P}_\tau(2n)$ we consider to deform it to a new path η in $\mathcal{P}_\tau(2n)$ so that

$$i_1(\gamma^m) = i_1(\eta^m), \quad \nu_1(\gamma^m) = \nu_1(\eta^m), \quad \forall m \in \mathbf{N}, \tag{3.10}$$

and that $(i_1(\eta^m), \nu_1(\eta^m))$ is easy enough to compute. This leads to finding homotopies $\delta : [0, 1] \times [0, \tau] \to \mathrm{Sp}(2n)$ starting from γ in $\mathcal{P}_\tau(2n)$ and keeping the end points of the homotopy always stay in a certain suitably chosen maximal subset of $\mathrm{Sp}(2n)$ so that (3.10) always holds. In fact, this set was first discovered in [Lon2] as the path connected component $\Omega^0(M)$ containing $M = \gamma(\tau)$ of the set

$$\Omega(M) = \{N \in \mathrm{Sp}(2n) \quad | \quad \sigma(N) \cap \mathbf{U} = \sigma(M) \cap \mathbf{U} \text{ and}$$
$$\nu_\lambda(N) = \nu_\lambda(M) \ \forall \lambda \in \sigma(M) \cap \mathbf{U}\}. \tag{3.11}$$

Here $\Omega^0(M)$ is called the *homotopy component* of M in $\mathrm{Sp}(2n)$.

In [Lon2]-[Lon4], the following symplectic matrices were introduced as *basic normal forms*:

$$D(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \qquad \lambda = \pm 2, \tag{3.12}$$

$$N_1(\lambda, b) = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}, \qquad \lambda = \pm 1, b = \pm 1, 0, \tag{3.13}$$

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad \theta \in (0, \pi) \cup (\pi, 2\pi), \tag{3.14}$$

$$N_2(\omega, b) = \begin{pmatrix} R(\theta) & b \\ 0 & R(\theta) \end{pmatrix}, \qquad \theta \in (0, \pi) \cup (\pi, 2\pi), \tag{3.15}$$

where $b = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ with $b_i \in \mathbf{R}$ and $b_2 \neq b_3$.

Splitting numbers possess the following properties:

Lemma 3.3. (cf. [Lon2] and Lemma 9.1.5 of [Lon4]) *Splitting numbers $S_M^\pm(\omega)$ are well defined, i.e., they are independent of the choice of the path $\gamma \in \mathcal{P}_\tau(2n)$ satisfying $\gamma(\tau) = M$ appeared in (3.9). For $\omega \in \mathbf{U}$ and $M \in \mathrm{Sp}(2n)$, splitting numbers $S_N^\pm(\omega)$ are constant for all $N \in \Omega^0(M)$.*

Lemma 3.4. (cf. [Lon2], Lemma 9.1.5 and List 9.1.12 of [Lon4]) *For $M \in \mathrm{Sp}(2n)$ and $\omega \in \mathbf{U}$, there hold*

$$S_M^\pm(\omega) = 0, \qquad if \ \omega \notin \sigma(M). \tag{3.16}$$

$$S_{N_1(1,a)}^+(1) = \begin{cases} 1, & if \ a \geq 0, \\ 0, & if \ a < 0. \end{cases} \tag{3.17}$$

For any $M_i \in \mathrm{Sp}(2n_i)$ with $i = 0$ and 1, there holds

$$S_{M_0 \diamond M_1}^\pm(\omega) = S_{M_0}^\pm(\omega) + S_{M_1}^\pm(\omega), \qquad \forall \omega \in \mathbf{U}. \tag{3.18}$$

We have the following

Theorem 3.5. (cf. [Lon3] and Theorem 1.8.10 of [Lon4]) *For any $M \in \mathrm{Sp}(2n)$, there is a path $f : [0,1] \to \Omega^0(M)$ such that $f(0) = M$ and*

$$f(1) = M_1 \diamond \cdots \diamond M_k, \tag{3.19}$$

where each M_i is a basic normal form listed in (3.12)-(3.15) for $1 \leq i \leq k$.

4   Proof of the main theorems

In this section, we give the proofs of Theorems 1.1 and 1.2 by using the techniques similar to those in [LoZ1].

Proof of Theorem 1.2. We prove the theorem by showing that: If the number of prime closed geodesics is finite, then there exist at least $[\frac{n}{2}] - 2$ closed geodesics possessing irrational average indices. Thus in the rest of this paper, we will assume the following:

(F) There are only finitely many prime closed geodesics $\{c_j\}_{1 \le j \le p}$ on (S^n, F).

Denote by $\{P_{c_j}\}_{1 \le j \le p}$ the linearized Poincaré maps of $\{c_j\}_{1 \le j \le p}$. Suppose $\{M_{c_j}\}_{1 \le j \le p}$ are the basic normal form decompositions of $\{P_{c_j}\}_{1 \le j \le p}$ in $\{\Omega^0(P_{c_j})\}_{1 \le j \le p}$ as in Theorem 3.5. Then by §1.8 [Lon4] we have

$$e(M_{c_j}) \le e(P_{c_j}), \qquad 1 \le j \le p. \tag{4.1}$$

Since the flag curvature K of (S^n, F) satisfies $\left(\frac{\lambda}{\lambda+1}\right)^2 < K \le 1$ by assumption, then every nonconstant closed geodesic must satisfy

$$i(c) \ge n - 1, \tag{4.2}$$

by Theorem 3 and Lemma 3 of [Rad4].

Now it follows from Theorem 2.2 of [LoZ1] (Theorem 10.2.3 of [Lon4]) and (4.1) that

$$i(c_j^{m+1}) - i(c_j^m) - \nu(c_j^m) \ge i(c_j) - \frac{e(P_{c_j})}{2} \ge 0, \quad 1 \le j \le p, \ \forall m \in \mathbf{N}. \tag{4.3}$$

Here the last inequality holds by (4.2) and the fact that $e(P_{c_j}) \le 2(n-1)$.

Note that we have $\hat{i}(c_j) > n - 1$ for $1 \le j \le p$ under the pinching assumption by Lemma 2 of [Rad5]. Hence by the common index jump theorem (Theorem 4.3 of [LoZ1], Theorem 11.2.1 of [Lon4]), there exist infinitely many $(N, m_1, \ldots, m_p) \in \mathbf{N}^{p+1}$ such that

$$i(c_j^{2m_j}) \ge 2N - \frac{e(M_{c_j})}{2} \ge 2N - (n-1), \tag{4.4}$$

$$i(c_j^{2m_j}) + \nu(c_j^{2m_j}) \le 2N + \frac{e(M_{c_j})}{2} \le 2N + (n-1), \tag{4.5}$$

$$i(c_j^{2m_j-m}) + \nu(c_j^{2m_j-m}) \le 2N - (i(c_j) + 2S^+_{M_{c_j}}(1) - \nu(c_j)), \quad \forall m \in \mathbf{N}. \tag{4.6}$$

$$i(c_j^{2m_j+m}) \ge 2N + i(c_j), \quad \forall m \in \mathbf{N}, \tag{4.7}$$

moreover $\frac{m_j \theta}{\pi} \in \mathbf{Z}$, whenever $e^{\sqrt{-1}\theta} \in \sigma(P_{c_j})$ and $\frac{\theta}{\pi} \in \mathbf{Q}$. In fact, the $m > 1$ cases in (4.6) and (4.7) follow from (4.3), other parts follow from Theorem 4.3 of [LoZ1] or Theorem 11.2.1 of [Lon4] directly. More precisely, by Theorem 4.1 of [LoZ1] (in (11.1.10) in Theorem 11.1.1 of [Lon4], with $D_j = \hat{i}(c_j)$, we have

$$m_j = \left(\left[\frac{N}{M\hat{i}(c_j)}\right] + \chi_j\right) M, \quad 1 \le j \le p, \tag{4.8}$$

where $\chi_j = 0$ or 1 for $1 \le j \le p$ and $M \in \mathbf{N}$ such that $\frac{M\theta}{\pi} \in \mathbf{Z}$, whenever $e^{\sqrt{-1}\theta} \in \sigma(M_{c_j})$ and $\frac{\theta}{\pi} \in \mathbf{Q}$ for some $1 \le j \le p$.

By Theorem 3.5, we have

$$M_{c_j} \approx N_1(1,1)^{\diamond p_{j,-}} \diamond I_2^{\diamond p_{j,0}} \diamond N_1(1,-1)^{\diamond p_{j,+}} \diamond G_j, \qquad 1 \le j \le p \tag{4.9}$$

for some nonnegative integers $p_{j,-}$, $p_{j,0}$, $p_{j,+}$, and some symplectic matrix G_j satisfying $1 \notin \sigma(G_j)$. By (4.9) and Lemma 3.4 we obtain

$$2S^+_{M_{c_j}}(1) - \nu_1(M_{c_j}) = p_{j,-} - p_{j,+} \geq -p_{j,+} \geq 1 - n, \quad 1 \leq j \leq p. \tag{4.10}$$

Using (4.2) and (4.10), the estimates (4.4)-(4.7) become

$$i(c_j^{2m_j}) \geq 2N - (n-1), \tag{4.11}$$

$$i(c_j^{2m_j}) + \nu(c_j^{2m_j}) \leq 2N + (n-1), \tag{4.12}$$

$$i(c_j^{2m_j - m}) + \nu(c_j^{2m_j - m}) \leq 2N, \quad \forall m \in \mathbf{N}. \tag{4.13}$$

$$i(c_j^{2m_j + m}) \geq 2N + (n-1), \quad \forall m \in \mathbf{N}. \tag{4.14}$$

By Lemma 2.4, for every $i \in \mathbf{N}$, there exist some $m, j \in \mathbf{N}$ such that

$$E(c_j^m) = \kappa_i, \quad \overline{C}_{2i + \dim(z) - 2}(E, c_j^m) \neq 0, \tag{4.15}$$

and by §2, we have $\dim(z) = n + 1$.

Claim 1. *We have the following*

$$m = 2m_j, \quad \text{if} \quad 2i + \dim(z) - 2 \in (2N, 2N + n - 1), \tag{4.16}$$

In fact, we have

$$\overline{C}_q(E, c_j^m) = 0, \quad \text{if} \quad q \in (2N, 2N + n - 1) \tag{4.17}$$

for $1 \leq j \leq p$ and $m \neq 2m_j$ by (4.13), (4.14) and Proposition 2.1. Thus in order to satisfy (4.15), we must have $m = 2m_j$.

It is easy to see that

$$^{\#}\{i : 2i + \dim(z) - 2 \in (2N, 2N + n - 1)\} = \left[\frac{n}{2}\right] - 1. \tag{4.18}$$

Claim 2. *There are at least $[\frac{n}{2}] - 1$ closed geodesics in $\mathcal{V}_\infty(S^n, F)$.*

In fact, for any N chosen in (4.4)-(4.7) fixed and $q \equiv 2i + \dim(z) - 2 \in (2N, 2N + n - 1)$, there exist some $1 \leq j_q \leq p$ such that c_{j_q} is $(2m_{j_q}, q)$-variationally visible by (4.15) and (4.16). Moreover, if $q_1 \neq q_2$, then we must have $j_{q_1} \neq j_{q_2}$. This holds by (4.15):

$$E(c_{j_{q_1}}^{2m_{j_{q_1}}}) = \kappa_{i_1} \neq \kappa_{i_2} = E(c_{j_{q_2}}^{2m_{j_{q_2}}}).$$

since κ_i are pairwise distinct by Lemma 2.3, where $q_k \equiv 2i_k + \dim(z) - 2$ for $k = 1, 2$. Hence the map

$$\Psi : (2\mathbf{N} + \dim(z) - 2) \cap (2N, 2N + n - 1) \to \{c_j\}_{1 \leq j \leq p}, \quad q \mapsto c_{j_q} \tag{4.19}$$

13

is injective. We remark here that if there are more that one c_j satisfy (4.15), we take any one of it. This proves $p \geq [\frac{n}{2}] - 1$. Since we have infinitely many N satisfying (4.4)-(4.7) and the number of prime closed geodesics is finite, we must have $[\frac{n}{2}] - 1$ closed geodesics in $\mathcal{V}_\infty(S^n, F)$.

We denote these closed geodesics by $\{c_j\}_{1 \leq j \leq [\frac{n}{2}] - 1}$, where $\{c_j\}_{1 \leq j \leq [\frac{n}{2}] - 1} \subset \mathrm{im}\Psi$.

Claim 3. *There are at least $[\frac{n}{2}] - 2$ closed geodesics in $\mathcal{V}_\infty(S^n, F)$ possessing irrational average indices.*

We prove the claim as the following: Let $D_j = \hat{i}(c_j)$ for $1 \leq j \leq p$. Then by the proof of Theorem 4.1 of [LoZ1] or Theorem 11.1.1 of [Lon4]), we can obtain infinitely many N in (4.4)-(4.7) satisfying the further properties:

$$\frac{N}{M\hat{i}(c_j)} \in \mathbf{N} \quad \text{and} \quad \chi_j = 0, \qquad \text{if} \quad \hat{i}(c_j) \in \mathbf{Q}. \tag{4.20}$$

Now suppose $\hat{i}(c_j) \in \mathbf{Q}$ and $\hat{i}(c_k) \in \mathbf{Q}$ hold for some distinct $1 \leq j, k \leq [\frac{n}{2}] - 1$. Then by (4.8) and (4.20) we have

$$\begin{aligned}
2m_j \hat{i}(c_j) &= 2\left(\left[\frac{N}{M\hat{i}(c_j)}\right] + \chi_j\right) M\hat{i}(c_j) \\
&= 2\left(\frac{N}{M\hat{i}(c_j)}\right) M\hat{i}(c_j) = 2N = 2\left(\frac{N}{M\hat{i}(c_k)}\right) M\hat{i}(c_k) \\
&= 2\left(\left[\frac{N}{M\hat{i}(c_k)}\right] + \chi_k\right) M\hat{i}(c_k) = 2m_k \hat{i}(c_k).
\end{aligned} \tag{4.21}$$

On the other hand, by (4.19), we have

$$\Psi(q_1) = j, \quad \Psi(q_2) = k, \qquad \text{for some} \quad q_1 \neq q_2. \tag{4.22}$$

Thus by (4.15) and (4.16), we have

$$E(c_j^{2m_j}) = \kappa_{q_1} \neq \kappa_{q_2} = E(c_k^{2m_k}). \tag{4.23}$$

Since $c_j, c_k \in \mathcal{V}_\infty(S^n, F)$, by Theorem 2.6 we have

$$\frac{\hat{i}(c_j)}{L(c_j)} = 2\sigma = \frac{\hat{i}(c_k)}{L(c_k)}. \tag{4.24}$$

Note that we have the relations

$$L(c^m) = mL(c), \quad \hat{i}(c^m) = m\hat{i}(c), \quad L(c) = \sqrt{2E(c)}, \qquad \forall m \in \mathbf{N}, \tag{4.25}$$

for any closed geodesic c on (S^n, F).

Hence we have

$$
\begin{aligned}
2m_j \hat{i}(c_j) &= 2\sigma \cdot 2m_j L(c_j) = 2\sigma L(c_j^{2m_j}) \\
&= 2\sigma\sqrt{2E(c_j^{2m_j})} = 2\sigma\sqrt{2\kappa_{q_1}} \\
&\neq 2\sigma\sqrt{2\kappa_{q_2}} = 2\sigma\sqrt{2E(c_k^{2m_k})} \\
&= 2\sigma L(c_k^{2m_k}) = 2\sigma \cdot 2m_k L(c_k) = 2m_k \hat{i}(c_k).
\end{aligned} \tag{4.26}
$$

This contradict to (4.21) and then we must have $\hat{i}(c_j) \in \mathbf{R} \setminus \mathbf{Q}$ or $\hat{i}(c_k) \in \mathbf{R} \setminus \mathbf{Q}$. Hence there is at most one $1 \leq j \leq [\frac{n}{2}] - 1$ such that $\hat{i}(c_j) \in \mathbf{Q}$, i.e., there are at least $[\frac{n}{2}] - 2$ closed geodesics in $\mathcal{V}_\infty(S^n, F)$ possessing irrational average indices. The proof of Theorem 1.2 now complete. ∎

Proof of Theorem 1.3. This is just a modification of the proof of Theorem 1.2.

Since the metric is bumpy, i.e., all the closed geodesics on (S^n, F) are non-degenerate, hence we have $1 \notin \sigma(P_c)$ for any closed geodesics c on (S^n, F). Thus in the decomposition (4.9), we have $p_{j,-} = p_{j,0} = p_{j,+} = 0$ for $1 \leq j \leq p$. Hence we obtain

$$
2S_{M_{c_j}}^+(1) - \nu_1(M_{c_j}) = p_{j,-} - p_{j,+} \geq 0, \qquad 1 \leq j \leq p. \tag{4.27}
$$

Using (4.2) and (4.27), the estimates (4.4)-(4.7) become

$$
\begin{aligned}
i(c_j^{2m_j}) &\geq 2N - (n-1), \tag{4.28} \\
i(c_j^{2m_j}) + \nu(c_j^{2m_j}) &\leq 2N + (n-1), \tag{4.29} \\
i(c_j^{2m_j-m}) + \nu(c_j^{2m_j-m}) &\leq 2N - (n-1), \quad \forall m \in \mathbf{N}. \tag{4.30} \\
i(c_j^{2m_j+m}) &\geq 2N + (n-1), \quad \forall m \in \mathbf{N}. \tag{4.31}
\end{aligned}
$$

Now the whole proof of Theorem 1.2 remains valid if we replace all the intervals $(2N, 2N + n - 1)$ there by the intervals $(2N - (n-1), 2N + n - 1)$. More precisely, by Lemma 2.4, for every $i \in \mathbf{N}$, there exist some $m, j \in \mathbf{N}$ such that

$$
E(c_j^m) = \kappa_i, \quad \overline{C}_{2i+\dim(z)-2}(E, c_j^m) \neq 0. \tag{4.32}
$$

Claim 4. *We have the following*

$$
m = 2m_j, \quad \text{if} \quad 2i + \dim(z) - 2 \in (2N - (n-1), 2N + n - 1), \tag{4.33}
$$

In fact, we have

$$
\overline{C}_q(E, c_j^m) = 0, \qquad \text{if} \quad q \in (2N - (n-1), 2N + n - 1) \tag{4.34}
$$

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for $1 \leq j \leq p$ and $m \neq 2m_j$ by (4.30), (4.31) and Proposition 2.1. Thus in order to satisfy (4.33), we must have $m = 2m_j$.

It is easy to see that

$$^\#\{i : 2i + \dim(z) - 2 \in (2N - (n-1),\, 2N + n - 1)\} = n - 2. \tag{4.35}$$

Thus there are at least $n - 3$ closed geodesics in $\mathcal{V}_\infty(S^n, F)$ possessing irrational average indices by the same proof as Claims 2 and 3 above. The proof of Theorem 1.3 is finished. ∎

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